CONTINUALITY OF SET OF BILIPSCHITZ CLASSES IN EUCLIDEAN SPACE

A. Magazinov

Introduction

This paper is devoted to studying biLipschitz equivalence of Delone sets.

Let M be a metric space with distance $d_M(x, y)$. Denote by $B_\rho(x)$ and $B^\circ_\rho(x)$ respectively the closed and the open balls with radius ρ centered at x. A set $\mathcal{A} \subset M$ is a *Delone set*, if for some $0 < r < R$ the following conditions hold.

- $B^\circ_r(x) \bigcap B^\circ_r(y) = \varnothing$ for every $x, y \in \mathcal{A}$.
- $\bigcup\limits_{x\in\mathcal{A}} B_R(x) = M$.

Two Delone sets $\mathcal{A} \subset M_1$ and $\mathcal{B} \subset M_2$ are *biLipschitz equivalent*, if there exist a real $\lambda \geqslant 1$ and a bijection $F : \mathcal{A} \to \mathcal{B}$ such that the inequality

$$\frac{1}{\lambda} d_{M_1}(x, y) \leqslant d_{M_2}(F(x), F(y)) \leqslant \lambda d_{M_1}(x, y)$$

holds for every $x, y \in \mathcal{A}$.

Map F for which such an inequality holds is called λ-*biLipschitz*.

The question about biLipschitz equivalence was raised by M. Gromov in [1]. In particular, the following problem was stated:

Given a metric space M determine if every two Delone sets $\mathcal{A}, \mathcal{B} \subset M$ are biLipschitz equivalent.

If $M = \mathbb{E}^1$ — a Euclidean line, then the answer is obviously positive. Also positive answers were obtained by P. Papasoglu (see [2]) for homogeneous trees, O. Bogopolsky (see [3]) for hyperbolic spaces $\mathbb{H}^d$ and K. Whyte (see [4]) for non-amenable spaces.

In case of Euclidean space $M = \mathbb{E}^d$ of dimension $d \geqslant 2$ D. Burago and B. Kleiner (see [5]) and independently C.McMullen (see [6]) proved the following result:

Theorem 1. *For every integer $d \geqslant 2$ there exists a Delone set $\mathcal{A} \subset \mathbb{E}^d$ which is not biLipschitz equivalent to the integer net $\mathbb{Z}^d$.*

In [5] theorem 1 is proved for $d = 2$, but the proof is easily generalized for every dimension $d \geqslant 2$. Therefore in $\mathbb{E}^d$ for every $d \geqslant 2$ there exist at least 2 biLipschitz classes.

The main result of this paper is

Theorem 2. *For every integer $d \geqslant 2$ the set of biLipschitz classes in $\mathbb{E}^d$ has cardinality continuum.*

Proof of theorem 2

Obtain the upper estimate for cardinality of the set of biLipschitz classes.

Use the following result of A. Garber (see [7, lemma 2])

Lemma 3. *Let $\mathcal{A} \subset \mathbb{E}^d$ be a Delone set. Then there exists a Delone set $\mathcal{D} \subset \mathbb{Z}^d$ such that $\mathcal{A}$ and $\mathcal{D}$ are biLipschitz equivlent.*

From lemma 3 follows that every biLipschitz class has at least one member among subsets of $\mathbb{Z}^d$. Therefore cardinality of the set of biLipscitz classes is at most cardinality of family of all subsets containing in $\mathbb{Z}^d$, i.e. continuum. The upper estimate proved.

To prove the lower estimate obtain a continuum family of pairwise non-equivalent Delone sets. These sets will be members of some special class.

From this point we consider only rectangular coordinates in $\mathbb{E}^d$. Parallelepipeds (cubes) with edges parallel to coordinate lines are called *coordinate*.

Let Q be a coordinate cube. Denote by $m(Q)$ its vertex with the least sum of coordinates.

Consider a tiling T of $\mathbb{E}^d$ into coordinate cubes whose edge lengths belong to $[1, L]$. The set $\mathcal{A} = \{m(Q) : Q \in T\}$ is obviously a Delone set. Delone sets obtained in such a way are called *L-special*.

Consider a map $G_{\mathcal{A}} : \mathcal{A} \to T$ sending each point $x \in \mathcal{A}$ to a cube $Q \in T$ such that $x = m(Q)$.

A point x of special Delone set $\mathcal{A}$ is *standard* if $G_{\mathcal{A}}(x)$ a unit cube and *exceptional* otherwise.

Introduce some notation. Let

$$\mathcal{P}_{MN} = \mathbb{Z}^d \cap ([0, MN] \times [0, N)^{d-1}),$$

$$\mathcal{P}^i_{MN} = \mathbb{Z}^d \cap ([iN, (i + 1)N) \times [0, N)^{d-1}) \text{ for } i = 0, 1, \dots, M - 1,$$

$$u = (0, 0, \dots, 0),$$

$$v = (MN, 0, 0, \dots, 0).$$

Call points $x, y \in \mathcal{P}_{MN}$ *corresponding* if $y - x = (N, 0, 0, \dots, 0)$.

Lemma 4. *Let $\lambda \geqslant 1$, $\varepsilon \in (0, \frac{1}{4})$ $a \in (0, 1)$. Then there exist $k > 0$ and $M_0 \in \mathbb{N}$ such that for every $M, N \in \mathbb{N}$ $M > M_0$ and for arbitrary λ-biLipschitz map $F : \mathcal{P}_{MN} \to \mathbb{E}^d$ at least one of the following statements hold:*

1. *There exist corresponding points x, y such that*

$$\frac{|F(y) - F(x)|}{|y - x|} > (1 + k)\frac{|F(v) - F(u)|}{|v - u|};$$

2. *There exists i such that number of pairs of corresponding points $x \in \mathcal{P}^i_{MN}$, $y \in \mathcal{P}^{i+1}_{MN}$ for which holds*

$$\frac{|F(y) - F(x) - \frac{1}{M}(F(v) - F(u))|}{\frac{1}{M}|F(v) - F(u)|} < \varepsilon,$$

   is at least aN^d.

Proof for $d = 2$ is in [5, Lemma 3.2]. Proof for an arbitrary d is obtained by a straightforward repeating the arguments of [5].

Lemma 5. *Let $I = [0, 1]$, $\alpha \in (0, \frac{1}{2})$, and let $P, Q \subset I^d$ be closed sets with a boundary being a finite polyhedron. If $P \cup Q = I^d$, $\operatorname{int} P \cap \operatorname{int} Q =$*

$\varnothing$ and also $\mathrm{Vol}_d(P) \geqslant \alpha$ *and* $\mathrm{Vol}_d(Q) \geqslant \alpha$ *then* $(d-1)$*-dimensional volume* $\mathrm{Vol}_{d-1}(\partial P \cap \partial Q) \geqslant \frac{\alpha}{2^{d-1}}$.

Proof. Denote by π the projection onto hyperplane
$x_1 = 0$.

Conduct the proof by induction over d.

Induction base: $d = 2$. If $\mathrm{Vol}_1(\pi(P) \cap \pi(Q)) \geqslant \frac{\alpha}{2}$ then statement of lemma is obviously true. Otherwise the following inequalities hold:

$$1 - \alpha \geqslant 1 - \mathrm{Vol}_2(Q) = \mathrm{Vol}_2(P) \geqslant \mathrm{Vol}_1(\pi(P)) - \mathrm{Vol}_1(\pi(P) \cap \pi(Q)).$$

Hence $\mathrm{Vol}_{d-1}(\pi(P)) < 1 - \frac{\alpha}{2}$.

Therefore there exists $t_P \in (0,1)$ such that $P \cap \{x_2 = t_P\} = \varnothing$. Similarly, there exists $t_Q \in (0,1)$ such that $Q \cap \{x_2 = t_Q\} = \varnothing$. It follows that projection of P, Q onto line $x_2 = 0$ is a segment $[0,1]$. Hence $\mathrm{Vol}_1(\partial P \cap \partial Q) \geqslant 1 > \frac{\alpha}{2}$ and for $d = 2$ statement is proved.

Induction step. Similarly to previous if $\mathrm{Vol}_{d-1}(\pi(P) \cap \pi(Q)) \geqslant \frac{\alpha}{2}$ statement of lemma is obvious. Otherwise $\mathrm{Vol}_{d-1}(\pi(P)) < 1 - \frac{\alpha}{2}$. Then every section of Q by a hyperplane $x_1 = t$ has a $(d-1)$-volume $\geqslant \frac{\alpha}{2}$. Similarly, every section of P by a hyperplane $x_1 = t$ has a $(d-1)$-volume $\geqslant \frac{\alpha}{2}$. By induction assumption, every section of $\partial P \cap \partial Q$ has a $(d-2)$-volume $\geqslant \frac{\alpha}{2^{d-1}}$, and the statement of lemma is now obvious.

Lemma 6. *Given $\lambda \geqslant 1$, $L \geqslant 1$ and rational $c > 1$ there exists a finite point set $\mathcal{B}_0$ and a parallelepiped $\Pi = \prod_{i=1}^{d} [0, b_i)$ $\;b_i \in \mathbb{N}$ such that:*

1. *$\mathcal{B}_0 \subset \Pi$.*
2. *There exists a tiling T_0 of Π into coordinate cubes with edges 1 and c such that $\{m(Q) : Q \in T_0\} = \mathcal{B}_0$.*
3. *For every Delone set $\mathcal{B}$ such that $\mathcal{B} \cap \Pi = \mathcal{B}_0$ and $(0, 0, \dots, 0, b_d) \in \mathcal{B}$ and for every λ-biLipschitz bijection $F : \mathcal{B} \to \mathcal{A}$ where $\mathcal{A}$ is L-special, the set $F(\mathcal{B}_0)$ has at least one exceptional point.*

Proof. Conduct the construction of $\mathcal{B}_0$ in 3 steps:

1. Choose ε $\;a$ which have the same meaning as in lemma 4; choose a parameter H_0.
2. Choose N and M.
3. Choose H fulfilling $H \geqslant H_0$ and construction of $\mathcal{B}_0$ itself.

Describe the construction beginning from the last step. Let $\varepsilon, a, M, N, H_0$ be already chosen on previous steps.

Take a parallelepiped

$$\Phi_{0,(0,0,\dots,0)} = [0,1)^{d-1} \times [0, M).$$

Consider its tiling into unit cubes. Colour these cubes checkerboardwise into black an white, starting with black.

Take in parallelepiped $\Phi_{0,(0,0,\ldots,0)}$ parallelepipeds

$$\Phi_{1,\frac{1}{N}\cdot(j_1,j_2,\ldots,j_{d-1},0)} = [0,\frac{1}{M})^{d-1}\times[0,M)+\frac{1}{N}\cdot(j_1,j_2,\ldots,j_{d-1},0)$$

where $j_i = 0,1,\ldots,N-1$. From this point colouring of $\Phi_{0,(0,0,\ldots,0)}$ will change only inside parallelepipeds of type $\Phi_{1,z}$. Divide each of these parallelepipeds into cubes with edge equal to $\frac{1}{M}$ and colour them checkerboardwise starting from black.

Continue the process. On ν-th step in each parallelepiped of type

$$\Phi_{\nu-1,z} = [0,\frac{1}{M^{\nu-1}})^{d-1}\times[0,M)+z$$

take the parallelepipeds

$$\Phi_{\nu,z+\frac{1}{N\cdot M^{\nu-1}}\cdot(j_1,j_2,\ldots,j_{d-1},0)} =$$
$$= [0,\frac{1}{M^{\nu}})^{d-1}\times[0,M)+z+\frac{1}{N\cdot M^{\nu-1}}\cdot(j_1,j_2,\ldots,j_{d-1},0).$$

From this point colouring will change only inside these parallelepipeds. Divide each of these parallelepipeds into cubes with edge equal to $\frac{1}{M^l}$ and colour them checkerboardwise starting from black.

Repeat while $\nu \leqslant \nu_0 = \lceil \log_{1+k}\lambda^2\rceil + 2$.

Note that if $\Phi_{0,(0,0,\ldots,0)}$ is divided into cubes with edge $\frac{1}{NM^{\nu_0}}$ then each of them is coloured in one colour — black or white. Call them *coloured cubes*

Make a homothety of parallelepiped $\Phi_{0,(0,0,\ldots,0)}$ together with colouring of coefficient H and center at origin. Choose H such that coloured cubes were taken into cubes that have integer edges and also could be divided into cubes with edge c. Inequality $H \geqslant H_0$ also must hold.

Images of black coloured cubes divide into unit cubes and images of white cubes — into cubes with edge c. The obtained tiling of $\Pi = H\cdot\Phi_{0,(0,0,\ldots,0)}$ denote by T_0. Let $\mathcal{B}_0 = \{m(Q) : Q \in T_0\}$.

Describe the second step. Let ε, a H_0 be chosen before, choose N and M.

Let $y-x = (0,0,\ldots,0,1)$. Choose N such that if $P = [0,\frac{1}{N}]^d$ then for every vector $\mathbf{e}$ fulfilling

$$|F(y)-F(x)-\mathbf{e})| < \varepsilon\cdot|\mathbf{e}|,$$

holds the inequality

$$|F(y')-F(x')-\mathbf{e}| < 2\varepsilon\cdot|\mathbf{e}|$$

if only $x'\in x+P$, $y'\in y+P$ and F is λ-biLipschitz.

Let P_1 P_2 be cubes with edge $\frac{H}{M^l}$ coloured on l-th step black and white respectively. Let each be divided into N^d equal cubes and let $Q_1\subset P_1$ $Q_2\subset P_2$ be such cubes. Choose M such that independently from choice of H holds true

$$\frac{\#(P_1\cap\mathcal{B}_0)}{\#(P_2\cap\mathcal{B}_0)} \geqslant \frac{1+c}{2}.$$

This inequality is obviously true if only

$$\mathrm{Vol}_d\left(Q_1 \cap (\cup_z \Phi_{l+1,z})\right) \leqslant \frac{1}{c-1}\,\mathrm{Vol}_d(Q_1) \text{ and}$$

$$\mathrm{Vol}_d\left(Q_2 \cap (\cup_z \Phi_{l+1,z})\right) \leqslant \frac{1}{c-1}\,\mathrm{Vol}_d(Q_2)$$

which is true for big enough M. Also take $M > M_0$ where M_0 comes from lemma 4 and $N|M$.

Describe the first step.

Let $a = \frac{3+c}{2+2c}$. Show that there exists a choice of ε and $H_0(\varepsilon)$ such that $\mathcal{B}_0$ constructed as before fulfilled the conditions of lemma 6.

Suppose that for every ε and H_0 there is a Delone set $\mathcal{B} \supset \mathcal{B}_0$ fulfilling the conditions of lemma 6 and λ-biLipschitz bijection $F : \mathcal{B} \to \mathcal{A}$ such that $F(\mathcal{B}_0)$ consists only of standard points.

Let $u = (0, 0, \dots, 0)$, $v = (0, 0, \dots, 0, HM)$. If the first case of statement of lemma 4 holds there exist corresponding points x, y such that $\frac{|F(y)-F(x)|}{|y-x|} > (1+k)\frac{|F(v)-F(u)|}{|v-u|}$. In this case instead of u, v consider a pair x, y and restriction of F to a subset of $\mathcal{B}_0$ contained in parallelepiped

$$x + [0, \frac{H}{M})^{d-1} \times [0, H).$$

Apply to this set all the arguments similarly as to $\mathcal{B}_0$. If such a substitution can be made $\lceil \log_{1+k} \lambda^2 \rceil + 2$ times, then from u, v we come to u', v' such that $\frac{|F(v')-F(u')|}{|v'-u'|} > \lambda^2 \frac{|F(v)-F(u)|}{|v-u|}$, which makes a contradiction to λ-biLipschitz property of F.

Therefore on some step we have the second case of lemma 4. Let the adjoint cubes for which this case holds have numbers $i \quad i+1$. Let also i-th cube be originally white and, respectively, $(i+1)$-th black. Let $\tilde{F} = G_{\mathcal{A}} \circ F$.

Let $\mathcal{C}$ be a set of points of i-th cube such that have distance at least $10\lambda L$ from its boundary, $\mathcal{C}'$ are all points of $i+1$-th cube. Using our assumptions obtain two inequalities involving $\mathrm{Vol}_{d-1}(\partial \tilde{F}(\mathcal{C}))$.

$$\mathrm{Vol}_{d-1}(\partial \tilde{F}(\mathcal{C})) \leqslant \beta_0 \cdot |\mathcal{C}|^{\frac{d-1}{d}},$$

$$\mathrm{Vol}_{d-1}(\partial \tilde{F}(\mathcal{C})) \geqslant \beta_2 \varepsilon^{-1} \cdot |\mathcal{C}|^{\frac{d-1}{d}}.$$

For small enough ε they contradict each other and that completes the proof of lemma 6.

Proving lemmas 7 and 8 H is assumed big enough depending on ε, i.e. $H \geqslant H_0(\varepsilon)$.

Lemma 7. *Inequality*

$$\mathrm{Vol}_{d-1}(\partial \tilde{F}(\mathcal{C})) \leqslant \beta_0 \cdot |\mathcal{C}|^{\frac{d-1}{d}}$$

holds true, where β_0 is a constant depending on d, λ, L and c (but not ε).

Proof. This inequality follows immediately from the fact that if $\tilde{F}(x)$ has common boundary with $\tilde{F}(\mathcal{C})$ then x is a point of i-th cube, that does not depend on $\mathcal{C}$. The number of such points does not exceed $\beta_1 \cdot |\mathcal{C}|^{\frac{d-1}{d}}$, hence $(d-1)$-volume of boundary of corresponding cubes does not exceed $\beta_0 \cdot |\mathcal{C}|^{\frac{d-1}{d}}$.

Let $s = 4\varepsilon \frac{1}{M}|F(v) - F(u)|$.

Let K be a real independent from ε and such that

$$\left(1 + \frac{(2K+2)^d - (2K)^d}{K^d}\right) \frac{4}{3+c} < \frac{8}{7+c}.$$

Take a full (in respect to inclusion relationship) packing of coordinate cubes with centers in $F(\mathcal{C})$ and edges equal to Ks. Let it consist of W cubes.

Denote by U a union of coordinate cubes with the same centers and edges equal to $2Ks$. Since the chosen packing is full all points of $F(\mathcal{C})$ are contained in U.

Let τ be a translation by vector $\frac{1}{M}(F(v)-F(u))$. Consider $\frac{s}{2}$-neighbourhood of $\tau(F(\mathcal{C}))$. Denote by $\mathcal{C}''$ the set of points of $\mathcal{A}$ that belong to this neighbourhood. Since the second case of lemma 4 assumed true and due to choice of a and M obtain:

$$|\mathcal{C}' \cap \mathcal{C}''| \geqslant a\frac{1+c}{2} \cdot |\mathcal{C}| = \frac{3+c}{4} \cdot |\mathcal{C}|.$$

But $\tilde{F}(\mathcal{C}' \cap \mathcal{C}'')$ is contained in the union of cubes with the same centers as $\tau(U)$ and edge equal to $(2K+2)s$, because for big H holds $s > 1$. Denote this union by U_1.

Note that $\mathrm{Vol}_d(U) \geqslant WK^d s^d$, $\mathrm{Vol}_d(U_1) \leqslant \mathrm{Vol}_d(U) + ((2K+2)^d - (2K)^d)Ws^d$. According to choice of K obtain

$$\mathrm{Vol}_d(\tilde{F}(\mathcal{C})) \leqslant \frac{4}{3+c}\,\mathrm{Vol}_d(U_1) \leqslant \frac{8}{7+c}\,\mathrm{Vol}_d(U).$$

Rewrite the last inequality as $\mathrm{Vol}_d(U \setminus \tilde{F}(\mathcal{C})) \geqslant \frac{c-1}{7+c}\,\mathrm{Vol}_d(U)$. Due to an estimate for $\mathrm{Vol}_d(U)$ already obtained,

$$\mathrm{Vol}_d(U \setminus \tilde{F}(\mathcal{C})) \geqslant \frac{c-1}{7+c} W K^d s^d.$$

Let $\mu \in (0,1)$ be such that $\mu + (1-\mu) \cdot \frac{c-1}{2^d(10+c)} < \frac{c-1}{2^d(7+c)}$. Note that μ does not depend on ε. Then in at least μW cubes of U set $\tilde{F}(\mathcal{C})$ occupies volume at most $\left(1 - \frac{c-1}{2^d(10+c)}\right) \cdot (2Ks)^d$. Call the cubes *marked*.

Lemma 8. *Suppose ε small enough, then in our assumptions on F*

$$\mathrm{Vol}_{d-1}(\partial \tilde{F}(\mathcal{C})) \geqslant \beta_1 \varepsilon^{-1} \cdot |\mathcal{C}|^{\frac{d-1}{d}},$$

where β_2 depends on d, λ, and c.

Proof. If $\varepsilon < \frac{1}{4K\lambda^2}$ and H is big enough then in every marked cube $\tilde{F}(\mathcal{C})$ occupies volume at least $\beta_3 \cdot (2Ks)^d$. Indeed, if $F(x)$ is a center of marked cube

then due to λ-biLipschitz property of F all points of $\mathcal{C} \cap B_{\frac{Ks}{\lambda}}(x)$ are taken inside this cube. For H big enough s is also big, then

$$\mathrm{Vol}_d(\tilde{F}(\mathcal{C} \cap B_{\frac{Ks}{\lambda}}(x))) = |\mathcal{C} \cap B_{\frac{Ks}{\lambda}}(x))| \geqslant 2\beta_3 \cdot (2Ks)^d,$$

and on the other hand, a part of volume of $\tilde{F}(\mathcal{C} \cap B_{\frac{Ks}{\lambda}}(x))$ not exceeding $(2Ks+2)^d - (2Ks)^d$ can be excluded from the marked cube. But for big enough s it does not exceed $\beta_3 \cdot (2Ks)^d$. Hence the inequality.

According to lemma 5 inside marked cubes $\partial\tilde{F}(\mathcal{C})$ has $(d-1)$-volume at least $\beta_4 s^{d-1}$ where β_4 depends on d, λ, and c.

Since cubes of packing do not intersect, no $8^d + 1$ cubes of U have a common point. Then for some β_5, depending on d, λ, and c holds

$$\mathrm{Vol}_{d-1}(\partial\tilde{F}(\mathcal{C})) \geqslant \beta_5 W s^{d-1}.$$

Since $F(\mathcal{C}) \subset U$ obtain $|\mathcal{C}| \leqslant W(2Ks+2)^d \leqslant \beta_6 W s^d$. Again H and s are assumed big enough. Therefore

$$\mathrm{Vol}_{d-1}(\partial\tilde{F}(\mathcal{C})) \geqslant \beta_7 |\mathcal{C}| s^{-1}.$$

Due to λ-biLipschitz property of F holds

$$\frac{1}{M}|F(v) - F(u)| \leqslant \beta_8 |\mathcal{C}|^{\frac{1}{d}}.$$

Using the definition of s obtain

$$s \leqslant \beta_9 \varepsilon |\mathcal{C}|^{\frac{1}{d}},$$

which together with the last inequality for $\mathrm{Vol}_{d-1}(\partial\tilde{F}(\mathcal{C}))$ implies the statement of lemma.

Lemma 9. *Given real $\lambda \geqslant 1$, $L \geqslant 1$, rational $c > 1$ and positive integer $j \in \mathbb{N}$ there exists a finite point set $\tilde{\mathcal{D}}$ and a parallelepiped $\Pi = \prod_{i=1}^{d} [0, b_i)$ where b_i are positive integer such that:*

1. *$\tilde{\mathcal{D}} \subset \Pi$.*
2. *There exists a tiling T_0 of Π into coordinate cubes with edges 1 and c such that $\{m(Q) : Q \in T_0\} = \tilde{\mathcal{D}}$.*
3. *For every Delone set $\mathcal{D}$ fulfilling $\mathcal{D} \cap \Pi = \tilde{\mathcal{D}}$ and $(0, 0, \ldots, 0, b_d) \in \mathcal{D}$ for every λ-biLipschitz bijection $F : \mathcal{D} \to \mathcal{A}$ with L-special Delone set $\mathcal{A}$, the set $F(\tilde{\mathcal{D}})$ contains at least j exceptional points.*

Proof. If $j = 1$ then the desired statement is exactly lemma 6. If $j > 1$ take a parallelepiped $\Pi(\lambda, L, c, j)$ with first $d-1$ edges equal to corresponding edges of $\Pi(\lambda, L, c, 1)$ and the last edge j times greater than the corresponding edge of $\Pi(\lambda, L, c, 1)$. Divide $\Pi(\lambda, L, c, j)$ into j parallelepipeds congruent to $\Pi(\lambda, L, c, 1)$.

Take in each of them a set congruent to $\tilde{\mathcal{D}}(\lambda, L, c, 1)$. Denote the obtained set by $\tilde{\mathcal{D}}(\lambda, L, c, j)$. Obviously, it fulfills the statement of lemma.

Return to the proof of theorem 2. Let $\{c_i\}_{i=1}^{\infty}$ be a sequence of rationals from $(1, 2]$, e.g. $c_i = 1 + \frac{1}{i}$.

In noataion of lemma 9 let $\mathcal{D}_1 = \tilde{\mathcal{D}}(1, 2, c_1, 1)$. By induction define

$$\mathcal{D}_j = \tilde{\mathcal{D}}(j, 2, c_j, \sum_{i=1}^{j-1} \#\mathcal{D}_i + 1).$$

Let $r_j = 100j \cdot \operatorname{diam}(\mathcal{D}_{j+1})$. Without loss of generality, let r_j be strictly increasing.

Let G be an additive group of rationals with denominator equal to some positive integer exponent of 2. From each class of $\mathbb{R}/G$ choose one number and for each chosen number take the sequence of digits after the point in its binary representation. Obtain a continuum set of non-confinal $(0, 1)$-sequences, i.e every two sequences have an infinite set of indices for which the corresponding members are different.

For every taken sequence $\alpha = \{\alpha_i\}_{i=1}^{\infty}$ construct a 2-special Delone set $\mathcal{D}_\alpha$ as follows. Take $\mathcal{D}_1$ so that the corresponding parallelepiped was coordinate with integer vertices. Further, if α has zero as j-th digit take a copy of $\mathcal{D}_{j+1}$ at $\lceil r_j \rceil$ to the right from $\mathcal{D}_j$; if α has unit as j-th digit then take a copy of $\mathcal{D}_{j+1}$ at $\lceil 100jr_j \rceil$ to the right from $\mathcal{D}_j$. Also corresponding to $\mathcal{D}_{j+1}$ parallelepiped should be coordinate with integer vertices. In addition, include into $\mathcal{D}_\alpha$ all points of $\mathbb{Z}^d$ which are outside all the parallelepipeds corresponding to $\mathcal{D}_j$. These points will be standard for $\mathcal{D}_\alpha$.

Prove that any two constructed sets are not biLipschitz equivalent.

Let there exist λ-biLipschitz bijection $F : \mathcal{D}_\alpha \to \mathcal{D}_\beta$. Lemma 10 states some property of this bijection.

Lemma 10. *For $j > 2\lambda$ there exists a point of $\mathcal{D}_\alpha$ in a copy of $\mathcal{D}_j$ (see construction of $\mathcal{D}_\alpha$) which is sent by F into some point of copy of $\mathcal{D}_j$ in $\mathcal{D}_\beta$.*

Proof. Indeed, since image of $\mathcal{D}_j$ contains many enough exceptional points, there is a point x of $\mathcal{D}_\alpha$ in a copy of $\mathcal{D}_j$ that is sent into an exceptional point, and, moreover, $F(x)$ does not belong to copies of $\mathcal{D}_i$ for $i < j$. If $F(x)$ belongs to copy of $\mathcal{D}_j$ then the proof is complete. Let $y = F(x)$ belong to copy of $\mathcal{D}_{j+k}$, $k > 0$. Consider the image of $\mathcal{D}_{j+k}$ under F^{-1} that is also λ-biLipschitz. Let $z \in \mathcal{D}_{j+k}$. Then

$$|F^{-1}(y) - F^{-1}(z)| \leqslant \lambda|y - z| \leqslant \lambda \operatorname{diam}(\mathcal{D}_{j+k}) < r_{j+k-1}.$$

Therefore all exceptional points in image of $\mathcal{D}_{j+k}$ under F^{-1} belong to copies of $\mathcal{D}_i$ for $i < j + k$, and it immediately implies a contradiction since there are many enough exceptional points in the image of $\mathcal{D}_{j+k}$.

Continue the proof of theorem 2.

Take $j > 2\lambda$ such that α and β differ in j-th digit. Without loss of generality, $\alpha_j = 0$ and $\beta_j = 1$. By lemma 10 there are x_j and x_{j+1} in copies of $\mathcal{D}_j$ and

$\mathcal{D}_{j+1}$ respectively in $\mathcal{D}_\alpha$ which are taken to points y_j and y_{j+1} of corresponding copies in $\mathcal{D}_\beta$. By construction of $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$,

$$100j \cdot \mathrm{diam}(\mathcal{D}_{j+1}) < |x_j - x_{j+1}| < (100j+2) \cdot \mathrm{diam}(\mathcal{D}_{j+1}), \text{ and}$$

$$10000j^2 \cdot \mathrm{diam}(\mathcal{D}_{j+1}) < |y_j - y_{j+1}| < (10000j^2+2) \cdot \mathrm{diam}(\mathcal{D}_{j+1}).$$

Therefore

$$\frac{|y_j - y_{j+1}|}{|x_j - x_{j+1}|} > 99j > \lambda.$$

A contradiction with λ-biLipschitz property of F makes proof of theorem 2 complete.

References

1. M. Gromov. Asymptotic invariants for infinite groups // London Mathematical Society Lecture Notes, vol. 182, Geometric group theory. eds. J. A. Niblo, M. A. Roller, J. W. S. Cassels, 1993.

2. P. Papasoglu. Homogeneous trees are bi-Lipschitz equivalent // Geom. Dedicata, vol. 54, 1995, 301-306.

3. O. V. Bogopolskii, Infinite commensurable hyperbolic groups are biLipschitz equivalent // Algebra and Logic, vol. 36, no. 3, 1997, 155-163.

4. K. Whyte, Amenability, bi-Lipschitz equivalence, and the von Neumann conjecture // Duke Math. J. vol. 99, 1999, 93-112.

5. D. Burago, B. Kleiner, Separated nets in Euclidean space and Jacobians of bi-Lipschitz maps //Geom. Funct. Anal. vol. 8, 1998, 273-282.

6. C. McMullen, Lipschitz maps and nets in Euclidean space // Geom. Funct. Anal. vol. 8, 1998, 304-314.

7. A. I. Garber, On equivalence classes of separated nets // Model. and Anal. of Inf. Syst., vol. 16, no. 2, 2009, 109-118 (in Russian).